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One-loop electron vertex in Yennie gauge

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Abstract. We derive a compact Yennie gauge representation for the off-shell one-loop electron-photon vertex, and discuss it properties. This expression is explicitly infrared finite, and it has proved to be extremely useful in multiloop calculations in the QED bound state problem.

1 Introduction

As is well known the physical results in quantum electrodynamics (as in any gauge theory) are gauge invariant, but the calculations themselves are gauge dependent. A proper choice of gauge may greatly facilitate calculations of radiative corrections in a specific problem.

In quantum electrodynamics gauge freedom is described by the transformation of the photon propagator

$$D_{\alpha\beta}(q) \to D_{\alpha\beta}(q) + q_\alpha\chi_\beta + \chi_\alpha q_\beta \ , \tag{1}$$

where χ_α are arbitrary functions of momentum q .

While the full gauge invariant sets of diagrams which describe the physical processes are gauge independent, the individual diagrams and the complexity of calculations strongly depend on the choice of gauge. The infrared safe Yennie gauge [1,2] defined by the photon propagator

$$D_{\alpha\beta}(q) = \frac{1}{q^2+i\varepsilon}\left(g_{\alpha\beta} + \frac{2q_\alpha q_\beta}{q^2}\right) \tag{2}$$

is particularly well suited for the bound state problems, where it greatly alleviates the notorious infrared difficulties specific for such kind of problems (see, e.g., [3–7]).

There is no infrared photon radiation in the bound state problems, and all infrared divergences should cancel in the final results. The most useful technically feature of the Yennie gauge, which is shares with the noncovariant Coulomb gauge, is that the infrared behavior of the individual diagrams is greatly improved in comparison with the infrared behavior of the diagrams in other covariant gauges. In particular many diagrams, which are infrared divergent in other relativistic gauges (Feynman, Landau, etc.), are infrared finite in the Yennie gauge. This feature of the Yennie gauge allows to perform explicitly covariant calculations without introducing an intermediate infrared photon mass, which is inevitable in other common relativistic gauges. Thus the Yennie gauge combines the nice infrared properties of the noncovariant Coulomb gauge (see, e.g., [8,9]) with the advantages specific to the explicitly Lorentz covariant gauges.

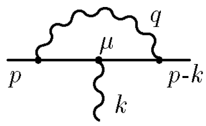

Fig. 1. One-loop vertex

The Yennie gauge is widely used in the bound state theory (see, e.g., [3–7], and references in [10]). In the framework of dimensional regularization one- and two-loop calculations in the Yennie gauge were discussed in [11–14]. The Yennie gauge was extensively used in our papers on the one- and two-loop radiative corrections to the bound state energy levels [6,7,15–18]. We have obtained a compact infrared soft integral representation for the renormalized one-loop vertex in the Yennie gauge, which turned out to be extremely useful in calculations. Below we will derive this representation for the Yennie gauge vertex, and discuss its main features.

2 Infrared finite bare vertex

General expression for the off-mass-shell one-loop vertex in the Yennie gauge (see Fig. 1) has the form

$$\Lambda_\mu(p,p-k) = \frac{\alpha}{4\pi}\int\frac{d^4q}{\pi^2 i} \times\frac{\gamma^\alpha(\hat p+\hat q+m)\gamma_\mu(\hat p+\hat q-\hat k+m)\gamma^\beta}{D(p+q)D(p+q-k)} \times D_{\alpha\beta}(q)\ , \tag{3}$$

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where

$$D(p) = p^2 - m^2 + i\varepsilon \ . \tag{4}$$

We would like to obtain an integral representation for the Yennie gauge vertex which is explicitly infrared finite. Ultraviolet divergences in the Yennie gauge should be subtracted as usual, and to this end it is convenient to have a simple expression for the ultraviolet divergent term. Let us first separate the ultraviolet divergent contributions. They are generated by the large integration momenta $q \to \infty$ in (3) when the Feynman and longitudinal parts of the numerator may be written as

$$\begin{aligned}
&\gamma^\alpha(\hat{p} + \hat{q} + m)\gamma_\mu(\hat{p} + \hat{q} - \hat{k} + m)\gamma_\alpha \\
&\quad \simeq \ q^2\gamma_\mu \ \simeq \ D(p + q - k)\gamma_\mu \ , \\
&\frac{2}{q^2} \ \hat{q}(\hat{p} + \hat{q} + m)\gamma_\mu(\hat{p} + \hat{q} - \hat{k} + m)\hat{q} \\
&\quad \simeq \ 2q^2\gamma_\mu \ \simeq \ 2D(p + q - k)\gamma_\mu \ .
\end{aligned} \tag{5}$$

We have represented the large terms in the numerator as coefficients before the electron denominator $D(p + q - k)$ in order to get rid of the k dependence in the ultraviolet divergent contributions.

The Yennie gauge vertex is infrared finite, and we are looking for such representation of the electron-photon vertex where all would be infrared divergences cancel already in the integrand. The most infrared singular terms in the integrand in (3) correspond to the terms in the numerator which do not contain the integration momentum q

$$\begin{aligned}
&\gamma^\alpha(\hat{p} + \hat{q} + m)\gamma_\mu(\hat{p} + \hat{q} - \hat{k} + m)\gamma^\beta \\
&\qquad \simeq \ \gamma^\alpha(\hat{p} + m)\gamma_\mu(\hat{p} - \hat{k} + m)\gamma^\beta \ .
\end{aligned} \tag{6}$$

Separating the ultraviolet and infrared divergent contributions we write the integral representation for the Yennie gauge vertex in (3) in the form

$$\begin{aligned}
&\Lambda_\mu(p, p - k) \\
&\quad = \frac{\alpha}{4\pi} \int \frac{d^4q}{\pi^2 i} \Big\{ 3\gamma_\mu H(0,1,1) \ + \ N_1 H(1,1,1) \\
&\qquad + \gamma^\alpha(\hat{p} + m)\gamma_\mu(\hat{p} - \hat{k} + m)\gamma^\beta \\
&\qquad \times \Big[g_{\alpha\beta} H(1,1,1) \ + \ 2q_\alpha q_\beta H(1,1,2) \Big] \Big\} \ ,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
N_1 = {} & 3q^2\gamma_\mu \ - \ 3D(q + p - k)\gamma_\mu + \ \gamma_\alpha\hat{q}\gamma_\mu(\hat{p} - \hat{k} + m)\gamma^\alpha \\
& + \gamma_\alpha(\hat{p} + m)\gamma_\mu\hat{q}\gamma^\alpha \\
& + 2\gamma_\mu(\hat{p} - \hat{k} + m)\hat{q} + 2\hat{q}(\hat{p} + m)\gamma_\mu \\
& - (2\hat{q}\gamma_\mu\hat{q} + q^2\gamma_\mu) \ ,
\end{aligned} \tag{8}$$

and

$$H(m, 1, n) \equiv \left\{ D^m(q + p - k) \, D(q + p) \, q^{2n} \right\}^{-1} \ . \tag{9}$$

All ultraviolet divergent contributions in (7) correspond to the numerators in (5), and are collected in the term with the denominator $H(0,1,1)$, which is independent of the transferred momentum k, and depends only on the external fermion momentum p. All potentially infrared divergent contributions to the integral generated by the Feynman and the longitudinal terms in the virtual photon propagator correspond to the numerator in (6), and are collected in the last term with the square brackets in the integrand in (7). All other terms in this integrand are explicitly infrared finite since they either contain an extra power of the virtual momentum q in the numerator, or respective denominators are less singular at small q.

The final integral representation will be written in terms of the combinations of external momenta which naturally arise when we combine the Feynman denominators. For example,

$$H(1,1,1) = \int_0^1 dx \int_0^1 dz \ \frac{2x}{\left[(q - xQ)^2 - x\Delta\right]^3} \ , \tag{10}$$

where

$$Q = -p + kz \ , \tag{11}$$

$$\Delta = m^2 - k^2 z(1 - xz) + 2pk(1 - x)z - p^2(1 - x) \ . \tag{12}$$

The ultraviolet divergent contribution generated by the term with the denominator $H(0,1,1)$ will be written in terms of the degenerate function Δ_0

$$\Delta_0 \equiv \Delta(z = 0) = m^2 x - D(p)(1 - x) \ . \tag{13}$$

After the shift of the integration variable $q \ \to \ q + xQ$ we obtain

$$\begin{aligned}
\Lambda_\mu(p, p - k) = {} & \frac{\alpha}{4\pi} \int \frac{d^4q}{\pi^2 i} \Big\{ 3\gamma_\mu \bar{H}(0,1,1) + \bar{N}_1 \bar{H}(1,1,1) \\
& + 2x^2 N_2 \bar{H}(1,1,2) + \gamma_\alpha(\hat{p} + m) \\
& \times \gamma_\mu(\hat{p} - \hat{k} + m)\gamma^\alpha \\
& \times \Big[\bar{H}(1,1,1) + \Big(\frac{q^2}{2} + 2x^2 Q^2 \Big) \bar{H}(1,1,2) \Big] \Big\} \\
= {} & \frac{\alpha}{4\pi} \int_0^1 dx \int_0^1 dz \Big\{ 3\gamma_\mu \cdot \Big[\ln \frac{\Lambda^2}{x\Delta_0} - 1 \Big] \\
& - \frac{\bar{N}_1}{\Delta} + 2N_2 \frac{x(1 - x)}{\Delta^2} + \gamma_\alpha(\hat{p} + m) \\
& \times \gamma_\mu(\hat{p} - \hat{k} + m)\gamma^\alpha \\
& \times \Big[-\frac{1}{\Delta} - \frac{1 - x}{\Delta} + \frac{2x(1 - x)Q^2}{\Delta^2} \Big] \Big\},
\end{aligned} \tag{14}$$

where Λ is the ultraviolet cutoff,

$$\begin{aligned}
\bar{N}_1 = {} & -3D(p - k)\gamma_\mu + x \Big[-6(p - k)Q\gamma_\mu \\
& + \gamma_\alpha\hat{Q}\gamma_\mu(\hat{p} - \hat{k} + m)\gamma^\alpha + \gamma_\alpha(\hat{p} + m)\gamma_\mu\hat{Q}\gamma^\alpha \\
& + 2\gamma_\mu(\hat{p} - \hat{k} + m)\hat{Q} + 2\hat{Q}(\hat{p} + m)\gamma_\mu \Big] \\
& - x^2 \Big(2\hat{Q}\gamma_\mu\hat{Q} + Q^2\gamma_\mu \Big),
\end{aligned} \tag{15}$$

$$\begin{aligned}
N_2 = {} & \hat{Q}(\hat{p} + m)\gamma_\mu(\hat{p} - \hat{k} + m)\hat{Q} - Q^2\gamma_\alpha(\hat{p} + m) \\
& \times \gamma_\mu(\hat{p} - \hat{k} + m)\gamma^\alpha,
\end{aligned} \tag{16}$$

and function $\bar{H}(m,1,n)$ is just the function $H(m,1,n)$ after the shift of the integration momentum $q \rightarrow q+xQ$.

The term with the numerator N_2 in (14) is the price we have to pay for the simple γ-matrix structure of the last term in the square brackets in (14). The point is that a more complicated matrix structure of the form $\hat{Q}(\ldots)\hat{Q}$ arises naturally after the shift $q \rightarrow q+xQ$. However, the trivial structure proportional to Q^2 greatly facilitates consideration of the would be infrared divergences, and we simply wrote infrared singular terms in the square brackets in (14) in a convenient form, and collected the compensating infrared safe terms in the numerator N_2.

The contributions generated by the terms $1/\Delta$ and x/Δ^2 in the last square brackets in (14) are infrared divergent on the mass shell at zero momentum transfer. Really, these terms behave as $1/x$ if $p^2 \rightarrow m^2$ and $k=0$, and thus are infrared divergent. In the Feynman gauge the vertex itself is also infrared divergent under these conditions but in the Yennie gauge infrared divergent contributions corresponding to the Feynman and longitudinal parts of the photon propagator cancel each other. In order to cancel these apparent infrared divergences we use the identity

$$\int_0^1 dx\,\frac{\partial}{\partial x}\left\{\frac{x(1-x)}{\Delta}\right\} = \int_0^1 dx\,\left\{\frac{1-2x}{\Delta}-\frac{x(1-x)Q^2}{\Delta^2}\right\}=0\ , \qquad (17)$$

which may be easily proved with the help of the relation $\partial\Delta/\partial x = Q^2$. Applying this identity we see that the infrared divergent contributions corresponding to the Feynman and longitudinal terms in the photon propagator cancel each other $-1-(1-x)+2(1-2x)=-3x$. Then the sum of the would be divergent contributions is reduced to the infrared safe form $-3xm^2/\Delta$, which is finite on the mass shell at zero momentum transfer.

After these transformations the Yennie gauge electron-photon vertex may be written in a compact form

$$\begin{aligned}&\Lambda_\mu(p,p-k)\\ &=\frac{\alpha}{4\pi}\int_0^1 dx\int_0^1 dz\Big\{3\gamma_\mu\left[\ln\frac{\Lambda^2}{m^2x^2}-1+\frac{D(p)}{\Delta_0}\right]\\ &-\frac{\bar{N}_1}{\Delta}-3\gamma_\alpha(\hat{p}+m)\gamma_\mu(\hat{p}-\hat{k}+m)\gamma^\alpha\frac{x}{\Delta}\\ &+2N_2\frac{x(1-x)}{\Delta^2}\Big\}. \qquad (18)\end{aligned}$$

3 Renormalization

As usual in QED to obtain the renormalized one-loop vertex from the expression in (18) we use mass shell subtraction at zero momentum transfer. However, due to absence of the infrared regularization the infrared finiteness of the subtraction term is not guaranteed, and we should first check that it is infrared finite. Consider asymptotic behavior of the vertex in (18) at $\rho_1 \equiv 1-p^2/m^2 \rightarrow 0$, $\rho_2 \equiv 1-(p-k)^2/m^2 \rightarrow 0$ and small but nonzero momentum transfer squared k^2. The largest contributions to the vertex in this regime have the form $k^2\ln\rho_i$, and they are generated by the term with N_2/Δ^2 in the integrand in (18). Throwing away all contributions which are at least linear in the virtualities of the electron lines, i.e., terms of the form ρ_1, ρ_2, $(\hat{p}-m)\simeq -m\rho_1/2$, and $(\hat{p}-k-m)\simeq -m\rho_2/2$, we obtain

$$N_2 \simeq 4\gamma_\mu\left[pk(-p^2+pkz)+(pk)^2z(-3+z)+p^2k^2z(1-z)\right]. \qquad (19)$$

It is easy to see that $k^2-2pk\rightarrow 0$ at $\rho_1\rightarrow 0$ and $\rho_2\rightarrow 0$. Then we substitute $2pk\rightarrow k^2$ in (19), and preserving only the leading in k^2 terms we have

$$N_2 \simeq 2[1+2z(1-z)]m^2k^2\gamma_\mu \rightarrow \frac{8}{3}\,m^2k^2\gamma_\mu\ , \qquad (20)$$

where we have effectively integrated over z on the right hand side. Integrating also over x we obtain

$$\begin{aligned}&\frac{\alpha}{4\pi}\int_0^1 dx\int_0^1 dz\frac{2x(1-x)N_2}{\Delta^2}\\ &\simeq\frac{4\alpha}{3\pi}\frac{k^2}{m^2}\gamma_\mu\int_0^1 dz\ln\frac{1}{(1-z)\rho_1+z\rho_2}\\ &\simeq\frac{4\alpha}{3\pi}\frac{k^2}{m^2}\gamma_\mu\ln\frac{1}{max(\rho_1,\rho_2)}\ . \qquad (21)\end{aligned}$$

This asymptotic behavior demonstrates that the Yennie gauge vertex admits subtraction on the mass shell without any additional infrared regularization. The only subtlety is that we first should put the momentum transfer squared to be zero, and only then go on the mass shell.

Let us calculate the subtraction constant. The numerator structures in (18) simplify at $k=0$ and $\hat{p}=m$:

$$-\bar{N}_1 \rightarrow 3x(2+x)m^2\gamma_\mu\ , \qquad (22)$$

$$-3x\gamma_\alpha(\hat{p}+m)\gamma_\mu(\hat{p}+m)\gamma^\alpha \rightarrow -12xm^2\gamma_\mu\ , \qquad (23)$$

$$N_2 \rightarrow 0\ ,\quad \Delta\rightarrow\Delta_0\rightarrow m^2x\ . \qquad (24)$$

Then the infrared finite subtraction constant may be easily calculated

$$\begin{aligned}\Lambda_\mu(m,m)&=\gamma_\mu\frac{3\alpha}{4\pi}\int_0^1 dx\\ &\times\left\{\ln\frac{\Lambda^2}{m^2x^2}-1+\frac{3x(2+x)}{x}-\frac{12x}{x}\right\}\\ &=\gamma_\mu\frac{3\alpha}{4\pi}\left(\ln\frac{\Lambda^2}{m^2}-\frac{1}{2}\right)\\ &\equiv\gamma_\mu\left(-1+Z_1^{-1}\right)\ . \qquad (25)\end{aligned}$$

The final expression for the unrenormalized Yennie gauge vertex is

$$\Lambda_\mu(p,p-k)=\gamma_\mu\left(-1+Z_1^{-1}\right)+\Lambda_\mu^R(p,p-k)\ . \qquad (26)$$

The ultraviolet and infrared finite renormalized electron-photon vertex in the Yennie gauge has the form

$$\Lambda_\mu^R(p,p-k)=\frac{\alpha}{4\pi}\int_0^1 dx\int_0^1 dz\left\{\frac{F_\mu^{(0)}}{\Delta_0}+\frac{F_\mu^{(1)}}{\Delta}+\frac{F_\mu^{(2)}}{\Delta^2}\right\}, \qquad (27)$$

where

$$F_\mu^{(0)} = 3\gamma_\mu D(p), \tag{28}$$

$$F_\mu^{(1)} = 3\gamma_\mu\Big[D(p-k) + (2-x)\Delta\Big] - x\Big[3\gamma_\alpha(\hat p + m) \times\gamma_\mu(\hat p - \hat k + m)\gamma^\alpha - 6(p-k)Q\gamma_\mu + \gamma_\alpha\hat Q\gamma_\mu(\hat p - \hat k + m)\gamma^\alpha + \gamma_\alpha(\hat p + m)\gamma_\mu\hat Q\gamma^\alpha + 2\gamma_\mu(\hat p - \hat k + m)\hat Q + 2\hat Q(\hat p + m)\gamma_\mu\Big] + x^2\Big(2\hat Q\gamma_\mu\hat Q + Q^2\gamma_\mu\Big), \tag{29}$$

$$F_\mu^{(2)} = 2x(1-x)\Big[\hat Q(\hat p + m)\gamma_\mu(\hat p - \hat k + m)\hat Q - Q^2\gamma_\alpha(\hat p + m)\gamma_\mu(\hat p - \hat k + m)\gamma^\alpha\Big]. \tag{30}$$

It is not difficult to check cancellation of the infrared finite renormalization constants Z_1 and Z_2 in the Yennie gauge. The explicit expression for the one-loop self-energy operator in the Yennie gauge is well known (see, e.g., [3, 6])

$$\Sigma(p) \ = \ \delta m \ + \ \left(-1 + Z_2^{-1}\right)(\hat p - m) \ + \ \Sigma^R(p) \ , \tag{31}$$

where the renormalized self-energy operator has the form

$$\Sigma^R(p) \ = \ \frac{\alpha}{4\pi}(\hat p - m)^2\int_0^1 dx \ \frac{-3\,\hat p\, x}{m^2 x - D(p)(1-x)} \ , \tag{32}$$

the mass renormalization is given by the expression

$$\delta m \ = \ \frac{3\alpha}{4\pi}\left(\ln\frac{\Lambda^2}{m^2} \ + \ \frac{1}{2}\right) , \tag{33}$$

and the wave function renormalization constant has the form

$$1 - Z_2^{-1} \ \equiv \ \Sigma'(m) \ = \ - \ \frac{3\alpha}{4\pi}\left(\ln\frac{\Lambda^2}{m^2} \ - \ \frac{1}{2}\right) . \tag{34}$$

It is easy to see that the Ward identity is satisfied, and the infrared finite renormalization constants Z_1 and Z_2 coincide

$$\Lambda_\mu(m,m) \ = \ - \ \Sigma'(m) \ , \qquad Z_1 \ = \ Z_2 \ . \tag{35}$$

4 Infrared and ultraviolet asymptotic behavior of the Yennie gauge vertex

The integral representation for the Yennie gauge vertex in (27) is most convenient for calculations of radiative corrections, and the usual representation of the vertex in terms of the Lorentz invariant form factors is neither necessary nor calculationally useful. However, quite often in the bound state problems one needs to treat separately the terms in the vertex which have different asymptotic behavior at small momentum transfer (see, e.g., [5,6]). All terms in the Yennie gauge vertex besides the anomalous magnetic moment contribution vanish at least as momentum transfer squared when $k^2 \to 0$. The anomalous magnetic moment contribution is linear in the momentum transfer, and it determines the asymptotic behavior of the Yennie gauge vertex at small momentum transfer. It is not difficult to identify the anomalous magnetic moment contribution in (27). All terms which contribute to the anomalous magnetic moment may be extracted from the term with the numerator $-\bar N_1$ in (14)

$$-x\gamma_\alpha\hat Q\gamma_\mu(\hat p - \hat k + m)\gamma^\alpha \to -2x(1-z)m\sigma_{\mu\nu}k^\nu \ , \tag{36}$$

$$-x\gamma_\alpha(\hat p + m)\gamma_\mu\hat Q\gamma^\alpha \to -2xzm\sigma_{\mu\nu}k^\nu \ , \tag{37}$$

$$2x^2\hat Q\gamma_\mu\hat Q \to 2x^2 m\sigma_{\mu\nu}k^\nu \ . \tag{38}$$

At small momentum transfer ($k \to 0$) on the mass shell ($p^2 = m^2$) the denominator $\Delta \to m^2 x$, and the sum of these terms generates the anomalous magnetic moment

$$- \ \frac{\alpha}{2\pi}\frac{\sigma_{\mu\nu}k^\nu}{2m} \ . \tag{39}$$

Separating the anomalous magnetic moment we write the renormalized one-loop vertex in the form

$$\Lambda_\mu^R(p,p-k) \ = \ \tilde\Lambda_\mu^R(p,p-k) \ - \ \frac{\alpha}{2\pi}\frac{\sigma_{\mu\nu}k^\nu}{2m} \ . \tag{40}$$

The scalar factors before the tensor structures in $\tilde\Lambda_\mu^R(p,p-k)$ depend only on the momentum transfer squared k^2 and the electron line virtuality $\rho = (m^2 - p^2)/m^2$. At small momenta transfer and near the mass shell all entries in the expression for $\tilde\Lambda_\mu^R(p,p-k)$ are either linear in k^2 and/or ρ, or are proportional to the projector $\hat p - m$ on the mass shell. At $k = 0$ we have

$$\tilde\Lambda_\mu^R(p,p) \ \simeq \ -\gamma_\mu\frac{3\alpha}{4\pi}\ \rho \ \simeq \ \gamma_\mu\frac{3\alpha}{2\pi}\ \frac{\hat p - m}{m} \ . \tag{41}$$

According to the Ward identity

$$\tilde\Lambda_\mu^R(p,p) \ = \ - \ \frac{\partial\Sigma^R(p)}{\partial p^\mu} \ , \tag{42}$$

the small momentum transfer behavior of the vertex is connected with the behavior of the self-energy operator near the mass shell. The mass operator in (32) near the mass shell $\hat p \to m$ is

$$\Sigma^R(p) \ \simeq \ - \ \frac{3\alpha}{4\pi}\ \frac{(\hat p - m)^2}{m} \ , \tag{43}$$

and it is easy to see that the Ward identity near the mass shell is satisfied.

We had used the Yennie gauge one-loop electron-photon vertex $\Lambda_\mu^R(p+q,p+q-k)$ from (27) as a subdiagram in two-loop calculations [16,17]. In such calculations not only the infrared but also the ultraviolet behavior of the one-loop vertex at $-q^2 \to \infty$ should be under control. The dominant logarithmic contribution to $\Lambda_\mu^R(q,q)$ is generated in this regime exclusively by the first term in the braces in (27)

$$\Lambda_\mu^R(q,q) \simeq -\frac{3\alpha}{4\pi}\gamma_\mu\ln\frac{-q^2}{m^2} \ . \tag{44}$$

All other contributions are nonlogarithmic. Ultraviolet behavior of the Yennie gauge vertex is no better than the ultraviolet behavior in the Feynman gauge, and really the contribution in (44) differs from the respective Feynman gauge expression only by a multiplicative factor 3. As is well known the ultraviolet vertex logarithm $\ln(-q^2/m^2)$ does not arise in the Landau gauge, which is the most convenient gauge for extracting the large ultraviolet logarithms.

Let us consider in more detail behavior of the different entries in the integrand in (27) at $-q^2 \to \infty$

$$F^{(1)}_\mu \simeq -3(1-x)^2 q^2\gamma_\mu + x(1+x)\left[q^2\gamma_\mu + 2\hat{q}\gamma_\mu\hat{q}\right] , \quad (45)$$

$$F^{(2)}_\mu \simeq 2x(1-x)q^2\left[q^2\gamma_\mu + 2\hat{q}\gamma_\mu\hat{q}\right] , \quad (46)$$

$$\Delta \simeq -(1-x)q^2 . \quad (47)$$

It is easy to see that in this regime the finite integral

$$\int_0^1 dx\left\{\frac{F^{(1)}_\mu}{\Delta} + \frac{F^{(2)}_\mu}{\Delta^2}\right\} \simeq \int_0^1 dx\left\{\frac{2}{-(1-x)q^2} + \frac{2(1-x)q^2}{(1-x)^2q^4}\right\} \times\left[q^2\gamma_\mu + 2\hat{q}\gamma_\mu\hat{q}\right] \quad (48)$$

is a sum of two integrals over the Feynman parameter x each of which diverges at $x \to 1$. In calculations of the two-loop radiative corrections we need to integrate this expression over the momentum q. Then it is often convenient and even necessary to consider the two terms in (48) separately. Note first that the factor

$$\left[q^2\gamma_\mu + 2\hat{q}\gamma_\mu\hat{q}\right] \simeq \left[q^2\gamma_\mu + \frac{2(-2)}{4}q^2\gamma_\mu\right] \quad (49)$$

vanishes after integration over q in the two-loop diagrams, if all other factors in the integrand depend only on q^2, but this is often not the case. Then one needs to avoid the spurious divergences at $x \to 1$ by rearranging different terms in (27) with the help of the identity in (17), which we already used to improve the infrared behavior. After transformations we obtain a slightly different representation for the renormalized vertex in the Yennie gauge

$$\Lambda^R_\mu(p,p-k) = \frac{\alpha}{4\pi}\int_0^1 dx\int_0^1 dz\left\{\frac{F^{(0)}_\mu}{\Delta_0} + \frac{\tilde{F}^{(1)}_\mu}{\Delta} + \frac{\tilde{F}^{(2)}_\mu}{\Delta^2}\right\} , \quad (50)$$

where

$$\tilde{F}^{(1)}_\mu = F^{(1)}_\mu + 2(1-2x)\left[q^2\gamma_\mu + 2\hat{q}\gamma_\mu\hat{q}\right] , \quad (51)$$

$$\tilde{F}^{(2)}_\mu = F^{(2)}_\mu - 2x(1-x)Q^2\left[q^2\gamma_\mu + 2\hat{q}\gamma_\mu\hat{q}\right] . \quad (52)$$

The transformation of the numerator structures in (51) does not change the infrared behavior of the vertex. Which of the representations (27) and (50) to use in calculations of the two-loop corrections depends on the nature of the two-loop diagram. For example, in calculation of the contributions of order $\alpha^2(Z\alpha)^5$ generated by the diagrams with the vertex insertions in the ultraviolet divergent skeleton diagrams (see e.g., diagrams (i,m,o) in [16, 17]) the representation in (50) is more convenient. On the other hand, the representation in (27) is more convenient for calculation of the contributions generated by the diagrams with the vertex insertions in the ultraviolet finite skeleton diagrams (see, e.g., diagrams (j,n) in [16,17]).

5 Discussion of results

In this paper we have described derivation and properties of a compact representation (27) for the off-shell one-loop electron-photon vertex in the Yennie gauge, which is convenient in multiloop calculations. In practice of such calculations one usually treats different terms in the integral representation of the electron-photon vertex separately. Hence, it is not sufficient to have a vertex with overall smooth asymptotic behavior but it is important to have well behaved individual terms in the integral representation for the vertex. We have specifically tailored these individual terms in such way that they are described by the well behaved finite integrals both in the infrared and ultraviolet regions. Respective integrals were briefly discussed above, because control of their behavior is absolutely crucial for successful applications when the one-loop vertex plays the role of a subdiagram in multiloop diagrams (see, e.g., [16,17]).

One- and two-loop renormalization in the Yennie gauge was considered earlier by G.Adkins [11–14]. In the framework of dimensional regularization he had obtained interesting integral representations for the Yennie gauge one-loop electron-photon vertex [11,13], which differ from the representations considered above. One-loop electron-photon vertex in an arbitrary gauge was also calculated in terms of elementary functions and dilogarithms in [19]. The goal of [19] was quite different from the aim of this work, namely to obtain a representation of the one-loop vertex suitable for nonperturbative generalizations. The vertex obtained in [19] was written as a sum of longitudinal and transverse parts which were further decomposed into sums of different spinor structures. Such representation is very convenient as a starting point for nonperturbative studies. However, separate analytic terms in [19] which explicitly contain the gauge parameter and are written in terms of the elementary functions and the Spence function are infrared singular. The would be infrared singularities of separate entries cancel only in the sum of these terms. This is precisely the feature we tried to avoid in the representation above cancelling the would be infrared singularities at the level of the integrands. This feature of the representation in (27) is extremely important for the applications to the bound state problems which we have in mind and partially discussed above. Direct comparison between the vastly different representations of the one-loop vertex in (27) and in [11,13,19] is greatly impeded by the complicated nature of these representations, it could be a subject of a special investigation which we would not attempt here.

The integral representations for the Yennie gauge electron-photon vertex in (27) and (50) were extensively used in our calculations of radiative corrections of order $\alpha^2(Z\alpha)^5$ to hyperfine splitting and Lamb shift [16,17], and in calculations of radiative-recoil corrections of order $\alpha(Z\alpha)^5(m/M)$ to the Lamb shift [18]. Soft infrared behavior in the Yennie gauge greatly facilitates the calculations. For example, almost all of the nineteen diagrams with two radiative photon insertions in the electron line and two external photon lines in [16,17] are infrared divergent in the Feynman gauge. In the Feynman gauge calculations of the radiative corrections of order $\alpha^2(Z\alpha)^5$ to hyperfine splitting and Lamb shift induced by these diagrams are greatly impeded by the infrared divergences though attainable [20–22]. Due to absence of the infrared divergences in the Yennie gauge and convenient form of the vertex in (27) and (50) the results of our calculations of the same contributions are about two orders of magnitude more accurate than the results in [20–23]. We hope that the Yennie gauge off-shell electron-photon vertex in (27) and (50) will find further useful applications.

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References

1. A. A. Abrikosov, Zh. Eksp. Teor. Fiz. **30**, 96 (1956) [Sov. Phys.-JETP **3**, 71 (1956)]
2. H. M. Fried, D. R. Yennie, Phys. Rev. **112**, 1391 (1958)
3. Y. Tomozawa, Ann. Phys. (NY) **128**, 491 (1980)
4. J. R. Sapirstein, E. A. Terray, D. R. Yennie, Phys. Rev. **D29**, 2290 (1984)
5. S. G. Karshenboim, V. A. Shelyuto, M. I. Eides, Yad. Fiz. **47**, 454 (1988) [Sov. J. Nucl. Phys. **47**, 287 (1988)]
6. M. I. Eides, S. G. Karshenboim, V. A. Shelyuto, Ann. Phys. (NY) **205**, 231 (1991)
7. M. I. Eides, S. G. Karshenboim, V. A. Shelyuto, Ann. Phys. (NY) **205**, 291 (1991)
8. G. S. Adkins, Phys. Rev. **D27**, 1814 (1983)
9. G. S. Adkins, Phys. Rev. **D34**, 2489 (1986)
10. M. I. Eides, H. Grotch, V. A. Shelyuto, preprint PSU/TH/226, hep-ph/0002158, February 2000, Physics Reports **342**, 63 (2001)
11. G. S. Adkins, Phys. Rev. **D39**, 3798 (1989)
12. G. S. Adkins, Phys. Rev. **D47**, 3647 (1993)
13. G. S. Adkins, M. Lymberopoulos, D. D. Velkov, Phys. Rev. **D50**, 4194 (1994)
14. G. S. Adkins, Yu. Zhang, Can. J. Phys. **76**, 333 (1998)
15. V. Yu. Brook, M. I. Eides, S. G. Karshenboim, V. A. Shelyuto, Phys. Lett. **216B**, 401 (1989)
16. M. I. Eides, V. A. Shelyuto, Pis'ma Zh. Eksp. Teor. Fiz. **61**, 465 (1995) [JETP Letters **61**, 478 (1995)]
17. M. I. Eides, V. A. Shelyuto, Phys. Rev. **A52**, 954 (1995)
18. M. I. Eides, H. Grotch, V. A. Shelyuto, Phys. Rev. **A63**, 052509 (2001)
19. A. Kizilersü, M. Reenders, M. R. Pennington, Phys. Rev. **D52**, 1242 (1995)
20. T. Kinoshita, M. Nio, Phys. Rev. Lett. **72**, 3803 (1994)
21. T. Kinoshita, M. Nio, Phys. Rev. D **53**, 4909 (1996)
22. K. Pachucki, Phys. Rev. Lett. **72**, 3154 (1994)
23. K. Pachucki, D. Leibfried, M. Weitz, A. Huber, W. König, T. W. Hänch, J. Phys. **B29**, 177 (1996); **B29**, 1573(E) (1996)